SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

A second-order equation, quite a mess,
With derivatives causing plenty of distress.
But Euler, Cauchy, and Lagrange helped evolve,
Elegant methods to solve and resolve!

As noted in the previous chapter, the order of a differential equation is defined as the order of the highest derivative of the unknown function that appears in the equation. Therefore, a second-order differential equation is one in which the second derivative of the unknown function occurs, and no derivative of higher order is present.

Second order differential equations have a variety of applications in science and engineering such as vibrations and electric circuits. There are a host of multi dimensional engineering models that incorporate second order differential equations including wave motion, flow mechanics, Maxwell's electro-magnetic equations and Schroedinger equation in Nuclear Physics.

7.1 2ND ORDER LINEAR ODE

The standard form of a 2nd Order ODE is:

$$y'' + p(x)y' + q(x)y = r(x) \qquad \text{It is linear in } y, y' \text{ and } y'' \tag{7.1.1}$$

If $r(x) = 0$, the ODE is homogeneous, else it is non-homogeneous. When the coefficients a and b are constant:

$$y'' + ay' + by = 0 \tag{7.1.2}$$

Choose $e^{\lambda x}$ as a solution and substitute in the homogeneous ODE.

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \implies \lambda^2 + a\lambda + b = 0 \implies \lambda = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

$$y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \qquad \text{(general solution to the homogeneous ODE)}$$

y_1, corresponding to λ_1, and y_2, corresponding to λ_2, are linearly independent and are called basis of solutions. The superposition principle also called the linearity principle, i.e., the homogeneous solution is a combination of y_1 and y_2 is true only for linear homogeneous ODE. The arbitrary constants c_1 and c_2 are determined from the initial conditions:

$$y(x_0) = k_0 \quad y'(x_0) = k_1$$

A particular solution is obtained if we assign specific values to c_1 and c_2.

7.2 LAGRANGE'S METHOD OF REDUCTION OF ORDER

Consider a linear homogeneous 2nd Order ODE in its standard form :

$$y'' + p(x)y' + q(x)y = 0 \tag{7.2.1}$$

If y_1 is a basis solution , we can find y_2 as follows:

$y_1 = e^{\lambda x}$

Let $y_2 = uy_1$

$y_2' = u' y_1 + uy_1'$

$y_2'' = u'' y_1 + 2u' y_1' + uy_1''$

Substituting,

$(u'' y_1 + 2u' y_1' + uy_1'') + p(u' y_1 + uy_1') + q(uy_1) = 0$

$y_1 u'' + (2y_1' + py_1)u' + (y_1'' + py_1' + qy_1)u = 0$

$u'' + u' \dfrac{2y_1' + py_1}{y_1} = 0$

Let $U = u'$

$U' + U \left(\dfrac{2y_1'}{y_1} + p \right) = 0$

$\dfrac{U'}{U} = - \left(\dfrac{2y_1'}{y_1} + p \right)$

$\displaystyle\int \dfrac{U'}{U} dx + \int \left(\dfrac{2y_1'}{y_1} \right) dx = -pdx$

$\ln |U| + 2\ln |y_1| = - \displaystyle\int pdx$

$\ln |Uy_1^2| = - \displaystyle\int pdx$

$Uy_1^2 = e^{\int -pdx}$

$$\boxed{U = \dfrac{1}{y_1^2} e^{\int -pdx}}$$

$$\boxed{u = \int U dx}$$

$$\boxed{y_2 = y_1 \int U dx}$$

$$\boxed{y_2 = y_1 \int \dfrac{1}{y_1^2} e^{\int -pdx} dx}$$

7.3   HOMOGENEOUS LINEAR ODE WITH CONSTANT COEFFICIENTS

$$\begin{cases} \text{Case 1: 2 Real Roots when} & a^2 - 4b > 0 \\ \text{Case 2: Double Root when} & a^2 - 4b = 0 \\ \text{Case 3: Complex Conjugate Roots when} & a^2 - 4b < 0 \end{cases}$$

Case 1: 2 Real Roots when $a^2 - 4b > 0$. The general solution is given by:

$$y_1 = e^{\lambda_1 x} \quad y_2 = e^{\lambda_2 x}$$

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right) \quad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

$$\boxed{y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}} \tag{7.3.1}$$

Case 2: $\boxed{\lambda_1 = -\frac{a}{2}, y_1 = e^{-\frac{ax}{2}}}$ Determine y_2 using the method of reduction of order.

$$y_1 = e^{\frac{-ax}{2}}$$

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{\int -p\,dx}\,dx = e^{\frac{-ax}{2}} \int \frac{1}{\left(e^{\frac{-ax}{2}}\right)^2} e^{\int -a\,dx}\,dx = e^{-\frac{ax}{2}} \int e^{ax} e^{-ax}\,dx = xe^{-\frac{ax}{2}}$$

$$y_h = c_1 e^{-ax/2} + c_2 x e^{-\frac{a}{2}x}$$

$$\boxed{y_h = (c_1 + c_2 x)e^{-ax/2}}$$

Case 3: $\boxed{\lambda = -\frac{a}{2} \pm \mathbf{i}w, w = \sqrt{|a^2 - 4b|}}$

$$y_1 = e^{\lambda_1 x} = e^{\left(-\frac{a}{2} + \mathbf{i}w\right)x} = e^{-\frac{ax}{2}} e^{\mathbf{i}wx}$$

$$y_2 = e^{\lambda_2 x} = e^{\left(-\frac{a}{2} - \mathbf{i}w\right)x} = e^{-\frac{ax}{2}} e^{-\mathbf{i}wx}$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) + i\left(x - \frac{x^3}{3!} - \frac{x^5}{5|} + \ldots\right) = \cos x + \mathbf{i}\sin x$$

$\boxed{e^{\mathbf{i}wx} = \cos wx + \mathbf{i}\sin wx}$ (de Moivre's theorem) and $\boxed{e^{\mathbf{i}\pi} = -1}$ (Euler's Identity)

Any real solution is a linear combination of the real and imaginary parts:

$\boxed{y = e^{-\frac{ax}{2}}\left(c_1 \cos wx + c_2 \sin wx\right)}$ c_1, c_2 are constants

7.4   EULER-CAUCHY EQUATIONS

The Euler-Cauchy equation is of the form:

$$\boxed{x^2 y'' + axy' + by = 0} \text{ where } a, b \text{ are constants} \tag{7.4.1}$$

Let $y = x^m \implies y' = mx^{m-1} \implies y'' = m(m-1)x^{m-2}$ and substituting

$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0 \implies m^2 + (a-1)m + b = 0$

$$\boxed{m = \frac{1}{2}(1-a) \pm \sqrt{\frac{1}{4}(a-1)^2 - b}}$$

Case 1: Roots are distinct. The basis solutions are :

$y_1(x) = x^{m_1} \quad y_2(x) = x^{m_2}$, the general solution is given by, $\boxed{y = c_1 x^{m_1} + c_2 x^{m_2}}$

Case 2: Double roots.

$$\boxed{y_1 = x^{\frac{1}{2}(1-a)}}$$

$$y'' + \frac{a}{x} y' + \frac{(1-a)^2}{4x^2} y = 0$$

Use method of reduction of order, $y_2 = uy_1$ and with $\boxed{p = \dfrac{a}{x}}$

$$\boxed{U = \frac{1}{y_1^2} e^{\int -pdx}} \quad \boxed{u = \int U dx} \quad \boxed{y_2 = y_1 \int U dx} \quad \boxed{y_2 = y_1 \int \frac{1}{y_1^2} e^{\int -pdx} dx}$$

$$\int pdx = \int \frac{a}{x} dx = a\ln x \implies e^{\int -pdx} = e^{-a\ln x} = e^{\ln x^{-a}} = x^{-a} = \frac{1}{x^a}$$

$$U = \frac{1}{y_1^2} \frac{1}{x^a} = \frac{1}{x^{1-a}} \frac{1}{x^a} = \frac{1}{x} \implies u = \int U dx = \int \frac{1}{x} dx = \ln x$$

$$y_2 = y_1 \int U dx = x^{\frac{1}{2}(1-a)} \ln x$$

$$\boxed{y_h = (c_1 + c_2 \ln x) x^{\frac{1}{2}(1-a)}} \quad c_1, c_2 \text{ are constants}$$

7.5 THE WRONSKIAN

Two solutions y_1 and y_2 are linearly dependent if their Wronkskian W is 0.

$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = 0$

Because if the solutions are dependent, $y_1 = ky_2$, where k is a constant

$\implies W(y_1, y_2) = y_1 y_2' - y_2 y_1' = ky_2 y_2' - y_2 ky_2' = 0$

The Wronksian is expressed as a Wronkski Determinant:

$$\boxed{W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \tag{7.5.1}$$

7.6 NON-HOMOGENEOUS ODE

Consider the following non-homogeneous ODE:

$$y'' + p(x)y' + q(x)y = r(x)$$

The complete solution is the sum of homogeneous (y_h) and particular (y_p)solutions.

$y(x) = y_h(x) + y_p(x)$ where $y_h = c_1 y_1 + c_2 y_2$ (general solution)

y_p is a solution of the non-homogeneous equation without any constants. A particular solution is obtained by assigning specific values to the constants. The Method of Undetermined Coeffcients is an approach to finding a particular solution to nonhomogeneous ODEs. If the term in $r(x)$ contains the following term, the choice for $y_p(x)$ is given by:

Term in r(x)	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$Kx^n \, (n = 0, 1, \ldots)$	$K_n x^n + K_{n-1} x^{n-1} + \ldots + K_1 x + K_0$
$k\cos wx$ or $k\sin wx$	$K\cos wx + M\sin wx$
$ke^{\alpha x}\cos wx$ or $ke^{\alpha x}\sin wx$	$e^{\alpha x}(K\cos wx + M\sin wx)$

7.7 Particular Solution by Variation of Parameters (Lagrange)

The particular solution for the standard form ODE is derived as follows:

$$y'' + p(x)y' + q(x)y = r(x)$$

Find a pair of functions $u_1(x)$ and $u_2(x)$ such that:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \implies y_p'(x) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

Set constraint, $\boxed{u_1'y_1 + u_2'y_2 = 0}$

$$y_p'(x) = u_1y_1' + u_2y_2'$$
$$y_p''(x) = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

Substituting,

$$(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = r$$
$$(y_1'' + py_1' + qy_1)u_1 + (y_2'' + py_2' + qy_2)u_2 + (u_1'y_1' + u_2'y_2') = r$$

Since y_1 and y_2 are solutions to the homogeneous ODE,

$$u_1'y_1' + u_2'y_2' = r$$

We now have the following simultaneous equations:

$$u_1'y_1 + u_2'y_2 = 0$$
$$u_1'y_1' + u_2'y_2' = r$$

Solving,

$$u_1' = -\frac{y_2 r}{y_1 y_2' - y_1' y_2} = -\frac{y_2 r}{W}$$
$$u_2' = -\frac{y_1 r}{y_1 y_2' - y_1' y_2} = -\frac{y_1 r}{W}$$

$$\boxed{y_p(x) = -y_1 \int \frac{y_2 r}{W}\,dx + y_2 \int \frac{y_1 r}{W}\,dx}$$

7.8 SymPy

Solve the following non-homogeneous ordinary differential equation:

$$2y(x) - 3\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = x^2$$

```python
import sympy as sp
from IPython.display import display, Math

# Variables and function
x = sp.symbols('x')
y = sp.Function('y')

# Nonhomogeneous ODE: y'' - 3 y' + 2 y = x^2
ode = sp.Eq(sp.diff(y(x), x, 2) - 3*sp.diff(y(x), x) + 2*y(x), x**2)
```

```python
10
11  # Solve full ODE
12  soln_full = sp.dsolve(ode)
13
14  # Extract complementary (homogeneous) and particular parts
15  # dsolve returns: Eq(y(x), C1*exp()+C2*exp() + particular)
16  rhs = soln_full.rhs
17  C1, C2 = sp.symbols('C1 C2')
18
19  # The homogeneous part is the expression containing C1, C2
20  soln_homo = rhs.expand().coeff(C1)*C1 + rhs.expand().coeff(C2)*C2
21
22  # The particular solution is the rest (remove terms with C1, C2)
23  soln_part = sp.simplify(rhs - soln_homo)
24
25  # Display results
26  print(sp.latex(ode))
27  print(sp.latex(soln_homo))
28  print(sp.latex(soln_part))
29  print(sp.latex(soln_full))
30
31  display(Math(r"\textbf{Homogeneous solution: } y_h = " + sp.latex(soln_homo)))
32  display(Math(r"\textbf{Particular solution: } y_p = " + sp.latex(soln_part)))
33  display(Math(r"\textbf{General solution: } y = " + sp.latex(rhs)))
34
35  print(sp.latex(soln_full))
36  soln_full
```

General solution:
$$y = C_1 e^x + C_2 e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$

Homogeneous solution:
$$y_h = C_1 e^x + C_2 e^{2x}$$

Particular solution:
$$y_p = \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$

General (or Full) solution:
$$y(x) = C_1 e^x + C_2 e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$