CHAPTER 8

Higher Order Ordinary Differential Equation

A higher order whispers through each change,
Derivatives weaving patterns wide and strange.
Roots shape motions—steady, wild, or deep—
In layered laws, the hidden forces sleep.

8.1 Higher Order Homogeneous ODE

The concepts of the 2nd Order ODE can be extended to higher order ODE which has the form:

$$\boxed{y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = r(x)}$$

For constant coefficients, $y = e^{\lambda x}$ yields:
$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \quad$ (characteristic equation)

For n distinct roots, there are n distinct basis solutions:

$$\boxed{y = c_1 e^{\lambda^1 x} + c_2 e^{\lambda^2 x} + \ldots + c_n e^{\lambda^n x}}$$

The Wronkskian is given by:

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ . & . & \ldots & . \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix} = E \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\ . & . & \ldots & . \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix}$$

Where $E = e^{(\lambda_1 + \lambda_2 + \ldots + \lambda_n)x}$

The determinant is known as the Vandermonde or Cauchy determinant. $W \neq 0$, if and only if, all the n roots are different.

If λ is a real root of order m, i.e., a real root of multiplicity m, the corresponding solutions are:

$$\boxed{e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x}, \ldots, x^{m-1} e^{\lambda x}}$$

Complex roots occur in conjugate pairs $\lambda = \gamma \pm iw$ since the coefficients of the ODE are real.

$$y_1 = e^{\gamma x}\cos(wx), \qquad y_2 = e^{\gamma x}\sin(wx).$$

If $\lambda = \gamma + iw$ is a complex double root (and hence $\gamma - iw$ also), then the corresponding linearly independent solutions are: $e^{\gamma x}\cos(wx), e^{\gamma x}\sin(wx), xe^{\gamma x}\cos(wx), xe^{\gamma x}\sin(wx)$. The corresponding general solution is: $y = e^{\gamma x}\boxed{[(A_1 + A_2 x)\cos(wx) + (B_1 + B_2 x)\sin(wx)]}$

For complex triple roots, one would obtain two more solutions: $\boxed{x^2 e^{\gamma x} cos\, wx}$ $\boxed{x^2 e^{\gamma x} sin\, wx}$

8.2 Higher Order Non-Homogeneous ODE

8.2.1 Method of Undetermined Coefficients

Use the method of undetermined coefficients with a small adjustment. If a term you would normally choose for $y_p(x)$ is already a solution of the homogeneous equation, multiply it by x^k, where k is the smallest positive integer that makes the new term no longer a solution of the homogeneous equation.

In practice, try:
$$cxe^{\lambda x}, \quad cx^2e^{\lambda x}, \quad \dots, \quad cx^ke^{\lambda x},$$
substitute into the ODE, and solve for c using the smallest k that works.

8.2.2 Method of Variation of Parameters

Consider the nth–order linear ODE in normalized form

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = r(x),$$

and let $y_1(x), \dots, y_n(x)$ be a fundamental set of solutions of the corresponding homogeneous equation. Let $W(x)$ denote their Wronskian.

To find a particular solution of the nonhomogeneous equation, we replace the constants in the homogeneous solution by functions and obtain the general variation-of-parameters formula:

$$\boxed{y_p(x) = \sum_{k=1}^{n} (-1)^{k+1} y_k(x) \int \frac{W_k(x)}{W(x)} r(x)\, dx}$$

where $W_k(x)$ is the determinant obtained from the Wronskian $W(x)$ by replacing its kth column with the vector $(0, 0, \dots, 0, 1)^T$.

In this construction,

- $W(x)$ ensures linear independence of the fundamental solutions;
- $W_k(x)$ comes from solving the system for the parameter derivatives using Cramer's rule;
- the alternating sign $(-1)^{k+1}$ reflects the cofactor expansion used in that determinant calculation.

Thus the formula generalizes the familiar second–order version to any order n, providing a systematic way to compute a particular solution once the homogeneous solutions are known.

8.3 Series Solutions of Homogeneous ODEs

Higher order linear ODEs with constant coefficients can be solved by algebraic methods as their solutions are often elementary functions which are known from calculus. For ODEs with variable coefficients the situation is complicated and their solutions are nonelementary special functions, e.g., Legendre and Bessel functions.

8.3.1 Power Series Method

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3^3 + \dots}$$

Compute $y', y'', \ldots, y^{(n)}$, substitute in the ODE and compute the coefficients of the powers of $x, x^2, x^3, \ldots, x^n$. Equate each of the coefficients to 0 to determine $a_0, a_1, a_2, \ldots, a_n$.

8.4 Existence of Power Series Solutions

Consider the following ODE:

$$\boxed{y'' + p(x)y' + q(x)y = r(x)}$$

If p, q, r have Taylor series representations (analytic) then every solution of the ODE can be represented by a power series in powers of $x - x_0$ with a positive radius of convergence R. A power series can be added, multiplied and differentiated term by term.

8.5 Classical Differential Equations

Legendre: $\boxed{(1-x^2)y'' - 2xy' + k(k+1)y = 0}$

Chebyshev: $\boxed{(1-x^2)y'' - xy' + k^2y = 0}$

Herimite: $\boxed{y'' - 2xy' + 2ky = 0}$

Laguerre: $\boxed{xy'' + (1-x)y' + ky = 0}$

where k is a constant

8.6 Legendre's Equation

$\boxed{(1-x^2)y'' - 2xy' + k(k+1)y = 0}$ k is a constant

Let $\boxed{y = \sum_{n=0}^{\infty} a_n x^n}$ and compute y, y', y'' to substitute in the above equation.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + k(k+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

Since $n(n-1)$ is 0 for $n = 0$ and $n = 1$, the lower indices start from 2 and 1.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + k(k+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

Let $n - 2 = m$ and use m as the index in the remaining terms as it is a dummy index:

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + k(k+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

a_0 and a_1 are arbitrary constants, the remaining constants are expressed in terms of these.

For $m = 0$,

$$2a_2 + k(k+1)a_0 = 0$$

$$\boxed{a_2 = -\frac{k(k+1)}{2!}a_0}$$

For $m = 1$,

$$6a_3 + [-2 + k(k+1)]a_1 = 0$$

$$\boxed{a_3 = -\frac{(k-1)(k+2)}{3!}a_1}$$

For $m \geq 2$,

$$(m+2)(m+1)a_{m+2} = [m(m-1) + 2m - k(k+1)]a_m = (m^2 + m - k^2 - k)a_m$$

$$\boxed{a_{m+2} = -\frac{(k-m)(k+m+1)}{(m+1)(m+2)}a_m} \qquad \text{for } m = 0, 1, 2, \ldots$$

Notice that the recurrence relation separates the coefficients into two independent groups: all *even* coefficients depend only on even ones, and all *odd* coefficients depend only on odd ones. Thus the full power series naturally splits into two independent series.

Independence of the solutions. A second-order linear ODE admits exactly two linearly independent solutions. Setting the initial data $(a_0 = 1, a_1 = 0)$ produces the even solution $y_1(x)$, while $(a_0 = 0, a_1 = 1)$ produces the odd solution $y_2(x)$. Even and odd functions cannot be constant multiples of one another, so these two solutions are necessarily independent.

Even-power series. Starting with a_0, the recurrence generates only even coefficients:

$$y_1(x) = 1 + a_2x^2 + a_4x^4 + \cdots$$

This series contains exclusively even powers of x and forms one solution of Legendre's equation.

Odd-power series. Starting with a_1, the recurrence generates only odd coefficients:

$$y_2(x) = x + a_3x^3 + a_5x^5 + \cdots$$

This series contains exclusively odd powers of x and forms the second, linearly independent solution.

General solution. Any solution of the Legendre equation can therefore be expressed as a linear combination of these two fundamental series:

$$y(x) = a_0\, y_1(x) + a_1\, y_2(x)$$

where a_0 and a_1 are arbitrary constants determined by boundary conditions.

8.6.1 Legendre Polynomials

When k is a nonnegative integer, the recurrence relation

$$a_{m+2} = -\frac{(k-m)(k+m+1)}{(m+1)(m+2)}a_m$$

eventually produces a factor $(k-m)$ in the numerator. Once $m = k$, this factor becomes zero, so:

$$a_{k+2} = a_{k+4} = a_{k+6} = \cdots = 0$$

This means the power series stops after finitely many terms — it becomes a polynomial. If k is even, the even series $y_1(x)$ terminates and becomes a polynomial of degree k. If k is odd, the odd series $y_2(x)$ terminates and becomes a polynomial of degree k.

These finite series are the *Legendre polynomials*, denoted by $P_k(x)$. Because they are polynomials, they are valid for all x (no convergence issues).

A common normalization is to choose the leading coefficient (the coefficient of x^k) as

$$a_k = \frac{(2k)!}{2^k(k!)^2}$$

To find the remaining coefficients, we use the recurrence in reverse:

$$a_m = -\frac{(m+1)(m+2)}{(k-m)(k+m+1)}\,a_{m+2} \qquad m < k$$

Example 1: $m = k-2$

$$a_{k-2} = -\frac{k(k-1)}{2(2k-1)}\,a_k = \frac{(2k-2)!}{2^k(k-1)!(k-2)!}$$

Example 2: $m = k-4$

$$a_{k-4} = \frac{(k-2)(k-3)}{4(2k-3)}\,a_{k-2} = \frac{(2k-4)!}{2^k\,2!\,(k-2)!(k-4)!}$$

In general, the coefficients are

$$\boxed{a_{k-2m} = (-1)^m \frac{(2k-2m)!}{2^k\,m!\,(k-m)!\,(k-2m)!}} \qquad m = 0, 1, 2, \ldots, \left\lfloor \frac{k}{2} \right\rfloor$$

Thus the Legendre polynomial can be written as

$$\boxed{P_k(x) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \frac{(2k-2m)!}{2^k\,m!\,(k-m)!\,(k-2m)!}\,x^{k-2m}} \qquad \lfloor k/2 \rfloor \text{ is floor of } k/2$$

8.7 Frobenius Method

Many important second–order ODEs have coefficients that are not analytic at the point of interest. A function f is said to be **analytic** at x_0 if it can be represented by a convergent power series in some neighborhood of x_0. That is, there exists $R > 0$ such that for all $|x - x_0| < R$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

and the series converges to $f(x)$.

In simple terms, analytic functions are infinitely differentiable and are *equal* to their Taylor series (not merely approximated by them). Examples include polynomials, e^x, $\sin x$, $\cos x$, and rational functions away from singularities. Non-analytic examples include $|x|$, functions with corners or cusps, and piecewise-defined functions with jumps.

Consider the ODE:

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0, \qquad \text{where } b(x) \text{ and } c(x) \text{ are analytic at } x = 0.$$

Although $b(x)$ and $c(x)$ are analytic, the coefficients $\frac{b(x)}{x}$ and $\frac{c(x)}{x^2}$ are *not* analytic at $x = 0$. Such equations have a *regular singular point* at $x = 0$, and can be handled using the Frobenius method.

Series Substitution

The Frobenius method seeks a solution of the form

$$\boxed{y(x) = x^r \sum_{m=0}^{\infty} a_m x^m} \qquad a_0 \neq 0,$$

where r may be real or complex.

Multiplying the ODE by x^2 gives

$$x^2 y'' + xb(x)\,y' + c(x)\,y = 0.$$

Expand

$$b(x) = \sum_{m=0}^{\infty} b_m x^m, \qquad c(x) = \sum_{m=0}^{\infty} c_m x^m,$$

and compute

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$$

$$y'(x) = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

Substituting into the ODE and collecting terms gives, for the lowest power of x,

$$\left[r(r-1) + b_0 r + c_0\right] a_0 = 0.$$

Because $a_0 \neq 0$, we obtain the **indicial equation**

$$\boxed{r^2 + (b_0 - 1)r + c_0 = 0.}$$

Let its roots be r_1 and r_2.

Resulting Solutions

The Frobenius method yields a fundamental set of solutions, depending on the relationship between r_1 and r_2.

Distinct roots not differing by an integer

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots),$$
$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots).$$

Double root

This happens when $(b_0 - 1)^2 = 4c_0$.

$$y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \cdots),$$
$$y_2(x) = y_1(x)\ln x + x^r(A_0 + A_1 x + A_2 x^2 + \cdots).$$

Roots differing by an integer

Assume $r_1 > r_2$ and $r_1 - r_2 \in \mathbb{Z}$.

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots),$$
$$y_2(x) = k\,y_1(x)\ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots),$$

where the constant k may be zero (so the logarithmic term may or may not appear depending on the recurrence).

In cases 2 and 3, the second independent solution can also be obtained by *reduction of order*.

8.8 Bessel's Equation

We study Bessel's differential equation:

$$\boxed{x^2y'' + xy' + (x^2 - \nu^2)\,y = 0} \qquad \nu \geq 0$$

Frobenius Ansatz

Ansatz is an assumed functional form for the solution, chosen so that it can be substituted into the equation to determine unknown coefficients or parameters.

Seek a solution of the form:

$$\boxed{y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}} \qquad a_0 \neq 0$$

Then, $y'(x) = \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1}$, $\quad y''(x) = \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}$

Substituting into the differential equation and collecting like powers of x yields:

$$\sum_{m=0}^{\infty}\Big(\big[(m+r)^2 - \nu^2\big]a_m + a_{m-2}\Big)x^{m+r} = 0$$

with the convention $a_{-1} = a_{-2} = \cdots = 0$. These coefficients do not exist in the original Frobenius series $\sum_{m=0}^{\infty} a_m x^{m+r}$, but appear when reindexing the x^{m+r+2} term. We set them to zero so that the recurrence relation applies uniformly for $m \geq 2$.

Hence the indicial equation (coefficient at $m = 0$) is:

$$\big[r(r-1) + r - \nu^2\big]a_0 = 0$$

$$(r^2 - \nu^2)a_0 = 0$$

$$\boxed{r = \pm\nu}$$

The general recurrence (for $m \geq 2$) is:

$$\boxed{a_m = -\frac{a_{m-2}}{(m+r)^2 - \nu^2}}$$

Case $r = \nu$

With $r = \nu$, the recurrence relation becomes

$$a_m = -\frac{a_{m-2}}{m(m+2\nu)}, \qquad m \geq 2.$$

To determine the odd-index coefficients, consider the case $m = 1$. Since $a_{-1} = 0$, the equation for $m = 1$ reduces to

$$((1+\nu)^2 - \nu^2)a_1 = (2\nu + 1)a_1 = 0$$

Because $2\nu + 1 \neq 0$, we conclude that:

$$a_1 = 0$$

The recurrence relation connects each odd coefficient only to the previous odd coefficient. Thus, from $a_1 = 0$, it follows inductively that all odd-index coefficients vanish:

$$a_1 = a_3 = a_5 = \cdots = 0.$$

Letting $m = 2k$, the recurrence relation becomes, for $k \geq 1$:

$$a_{2k} = -\frac{a_{2k-2}}{2k(2k+2\nu)}$$

The first few coefficients are:

$$a_2 = -\frac{a_0}{2^2(\nu+1)}, \qquad a_4 = \frac{a_0}{2^4\,2!\,(\nu+1)(\nu+2)}, \quad \dots$$

Repeatedly applying the recurrence yields the closed form:

$$\boxed{a_{2k} = \frac{(-1)^k\, a_0\, \Gamma(\nu+1)}{2^{2k} k!\, \Gamma(\nu+k+1)}} \qquad k \geq 0.$$

Here $\Gamma(n) = (n-1)!$ when n is a positive integer.

Thus the Frobenius solution corresponding to $r = \nu$ is

$$y_1(x) = a_0\, x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(\nu+1)}{2^{2k} k!\, \Gamma(\nu+k+1)} x^{2k}$$

Choosing the normalization:

$$a_0 = \frac{1}{2^\nu\, \Gamma(\nu+1)}$$

yields the standard Bessel function of the first kind:

$$\boxed{J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}}$$

Case $r = -\nu$

With $r = -\nu$, the recurrence relation becomes:

$$a_m = -\frac{a_{m-2}}{m(m-2\nu)}, \qquad m \geq 2$$

As before, the coefficient a_{-1} is taken to be zero, and the equation for $m = 1$ gives:

$$((1-\nu)^2 - \nu^2)a_1 = (1-2\nu)a_1 = 0,$$

so we take $a_1 = 0$. Hence, all odd-index coefficients vanish:

$$a_1 = a_3 = a_5 = \cdots = 0.$$

Letting $m = 2k$, the recurrence for the even-index coefficients is:

$$a_{2k} = -\frac{a_{2k-2}}{2k\,(2k-2\nu)}, \qquad k \geq 1.$$

Iterating this relation yields:

$$\boxed{a_{2k} = \frac{(-1)^k\, a_0\, \Gamma(1-\nu)}{2^{2k} k!\, \Gamma(1-\nu+k)}} \qquad k \geq 0$$

Thus the Frobenius solution corresponding to $r = -\nu$ is:

$$y_2(x) = a_0\, x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(1-\nu)}{2^{2k} k!\, \Gamma(1-\nu+k)} x^{2k}$$

A convenient normalization is:

$$a_0 = \frac{1}{2^{-\nu}\Gamma(1-\nu)} = 2^{\nu}\frac{1}{\Gamma(1-\nu)}$$

which yields the series expansion for the Bessel function $J_{-\nu}(x)$:

$$\boxed{J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(1-\nu+k)}\left(\frac{x}{2}\right)^{2k-\nu}}$$

When ν is not an integer, the two solutions $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent. For integer ν, the second independent solution involves a logarithmic term and leads to the Neumann function $Y_\nu(x)$.

8.8.1 Bessel functions for real order ν

To obtain a normalized solution for the Bessel function $J_\nu(x)$, we choose:

$$a_0 = \frac{1}{2^{\nu}\Gamma(\nu+1)}$$

The Gamma function

The Gamma function is defined for $\nu > -1$ by:

$$\Gamma(\nu+1) = \int_0^{\infty} e^{-t}t^{\nu}\,dt$$

Integrating by parts yields the functional identity:

$$\Gamma(\nu+1) = \nu\Gamma(\nu)$$

For positive integers n,

$$\Gamma(n+1) = n!$$

Thus the Gamma function extends the factorial to non-integer values.

Even coefficients

From the Frobenius method we obtain:

$$a_{2m} = \frac{(-1)^m a_0\Gamma(\nu+1)}{2^{2m}m!\Gamma(\nu+m+1)}, \qquad m \geq 0$$

The Bessel function $J_\nu(x)$

Substituting the coefficients into the Frobenius series gives:

$$J_\nu(x) = x^{\nu}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu}m!\Gamma(\nu+m+1)}$$

Equivalently,

$$\boxed{J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+m+1)}\left(\frac{x}{2}\right)^{2m+\nu}}$$

The function $J_\nu(x)$ is called the *Bessel function of the first kind of order* ν.

Useful identities

$[x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x)$	$[x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x)$
$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$	$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)$
$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$	$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

8.8.2 General solution

For non-integer ν, the functions $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent, and the general solution of Bessel's equation is:

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

When $\nu = n$ is an integer, $J_{-n}(x)$ is not independent of $J_n(x)$; in this case the second independent solution is the *Neumann function* $Y_n(x)$, which contains a logarithmic term.

8.8.3 Bessel functions of the second kind, $Y_0(x)$

For $\nu = 0$, Bessel's equation becomes:

$$xy'' + y' + xy = 0$$

The indicial equation has a double root $r_1 = r_2 = 0$, so the second linearly independent solution must be of the form:

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} A_m x^m$$

Differentiating,

$$y_2'(x) = J_0'(x) \ln x + \frac{J_0(x)}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$
$$y_2''(x) = J_0''(x) \ln x + \frac{2J_0'(x)}{x} - \frac{J_0(x)}{x^2} + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-2}$$

Substituting y_2, y_2', y_2'' into the differential equation gives:

$$(xJ_0'' + J_0' + xJ_0) \ln x + 2J_0'(x) + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

Since J_0 satisfies Bessel's equation,

$$xJ_0'' + J_0' + xJ_0 = 0$$

so the $\ln x$ term disappears.

Series expansions

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$$
$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m 2m\, x^{2m-1}}{2^{2m}(m!)^2} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!(m-1)!}$$

Substitute these into the earlier identity:

$$\sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!(m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

Solving for the coefficients

The constant term x^0 appears only in the middle series, so:

$$A_1 = 0$$

Comparing even powers of x (x^{2s}):

$$(2s+1)^2 A_{2s+1} + A_{2s-1} = 0, \qquad s = 0, 1, 2, \ldots$$

Since $A_1 = 0$, we obtain:

$$A_3 = A_5 = A_7 = \cdots = 0$$

Comparing odd powers of x:

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!\, s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0$$

This yields the closed-form expression:

$$A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right), \qquad m = 1, 2, \ldots$$

Define the harmonic number

$$h_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

The second solution is therefore:

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m}$$

J_0 and y_2 are linearly independent on $x > 0$. The standard form of the second Bessel solution is defined by

$$Y_0(x) = \frac{2}{\pi} \Big(y_2(x) + (\gamma - \ln 2)\, J_0(x) \Big)$$

where,

$$\gamma = \lim_{m \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} - \ln m\right) = 0.57721566490\ldots$$

is the Euler–Mascheroni constant.

Thus the Bessel function of the second kind of order 0 is:

$$\boxed{Y_0(x) = \frac{2}{\pi} \left[J_0(x) \left(\ln \frac{x}{2} + \gamma \right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m} \right]}$$

8.8.4 Bessel functions of the second kind, $Y_\nu(x)$

For general real ν, the second solution is defined (for non-integer ν) by:

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

For integer n,

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

which produces the required logarithmic term.

Thus the general solution of Bessel's equation for $x > 0$ is:

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

8.9 SymPy

```
# Solving Bessel's equation with SymPy
# Demonstrates symbolic solution for general order nu, and a concrete example for nu=1.
import sympy as sp

# symbols and function
x, nu = sp.symbols('x nu')
y = sp.Function('y')

# General Bessel's equation: x^2 y'' + x y' + (x^2 - nu^2) y = 0
ode_general = sp.Eq(x**2*sp.diff(y(x), x, 2) + x*sp.diff(y(x), x) + (x**2 - nu**2)*y(x), 0)
print(sp.latex(ode_general))

# Solve symbolically (returns solution in terms of BesselJ and BesselY)
sol_general = sp.dsolve(ode_general)
sol_general_simpl = sp.simplify(sol_general.rhs)  # RHS is the general solution expression

# Concrete example: nu = 1 (order 1 Bessel equation)
ode_nu1 = sp.Eq(x**2*sp.diff(y(x), x, 2) + x*sp.diff(y(x), x) + (x**2 - 1)*y(x), 0)
sol_nu1 = sp.dsolve(ode_nu1)

# Example with initial conditions: y(1)=1, y'(1)=0 for nu=1
ics = {y(1): 1, sp.diff(y(x), x).subs(x, 1): 0}
sol_nu1_ics = sp.dsolve(ode_nu1, ics=ics)

# Show results
print("General solution (order 'nu'):\n", sol_general_simpl, "\n")
print("Solution for nu = 1:\n", sol_nu1.rhs, "\n")
print(sp.latex(sol_nu1.rhs))
print("Solution for nu = 1 with y(1)=1, y'(1)=0:\n", sol_nu1_ics.rhs, "\n")

# Also explicitly show the independent basis functions
C1, C2 = sp.symbols('C1 C2')
basis = sp.Matrix([sp.besselj(nu, x), sp.bessely(nu, x)])
print("Fundamental solutions (Bessel J and Y):\n", basis)

# Return objects for inspection if desired
sol_general, sol_nu1, sol_nu1_ics, basis
```

$$x^2 \frac{d^2}{dx^2} y(x) + x \frac{d}{dx} y(x) + \left(-\nu^2 + x^2\right) y(x) = 0$$

$$\frac{\left(Y_2(1) - Y_0(1)\right) J_1(x)}{J_1(1) Y_2(1) + J_0(1) Y_1(1) - J_1(1) Y_0(1) - J_2(1) Y_1(1)} + \frac{\left(-J_2(1) + J_0(1)\right) Y_1(x)}{J_1(1) Y_2(1) + J_0(1) Y_1(1) - J_1(1) Y_0(1) - J_2(1) Y_1(1)}$$