Chapter 11

System of Ordinary Differential Equations

A system of ordinary differential equations (ODEs) is a set of differential equations that involve multiple dependent variables, each of which is a function of the same independent variable. In other words, it is a set of equations that describes how the rates of change of several variables depend on their current values and possibly the values of other variables.

Systems of ODEs are used to model a wide range of physical, biological, and engineering systems, such as chemical reactions, population dynamics, control systems, and many others. Solving or analyzing systems of ODEs is often challenging and requires a combination of analytical and numerical methods, such as numerical integration, linear algebra, and phase plane analysis.

Consider the following genearal system of ODE:

$$
\begin{aligned}
\frac{dy_1}{dt} &= f_1(t, y_1, y_2, \cdots y_n) \\
\frac{dy_2}{dt} &= f_2(t, y_1, y_2, \cdots y_n) \\
&\vdots \\
\frac{dy_n}{dt} &= f_n(t, y_1, y_2, \cdots y_n)
\end{aligned}
$$

Now consider the following linar system of ODE:

$$
\begin{aligned}
y_1' &= a_{11}(t)y_1 + \cdots + a_{1n}(t)y_n + g_1(t) \\
&\vdots \\
y_n' &= a_{n1}(t)y_1 + \cdots + a_{nn}(t)y_n + g_n(t)
\end{aligned}
$$

In a vector form,

$$y' = Ay + g$$

If $g = 0$, the system is homogeneous and we have:

$$y' = Ay$$

Consider the solution:

$$y = xe^{\lambda t}$$
$$y' = \lambda xe^{\lambda t} = Ay = Axe^{\lambda t}$$

We arrive at the Eigengvalue problem:

$$Ax = \lambda x$$

$$y_1' = a_{11}y_1 + a_{12}y_2$$
$$y_2' = a_{21}y_1 + a_{22}y_2$$

$$\frac{dy_2}{dy_1} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$$

A single ODE has a solution of the form $y = ce^{\lambda t}$
$y' = x\lambda e^{\lambda t}$

In vector form $Ax = \lambda x$ determine the eigenvalues and the eigenvector

$$det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22-\lambda} \end{vmatrix} = 0$$

The general solution is given by:
$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

11.1 Qualitative Analysis, Critical Point & Stability

For a qualitative analysis of a system of ODE, we use phase portrait which is a graphical representation of the behavior of a dynamical system over time. In particular, it shows the possible trajectories that a system can take in its **phase space,** which is a space of all possible states of the system. It is desirable that physical systems be stable, i.e., a small change at some instant causes only a small change in the behavior of the system at later times.

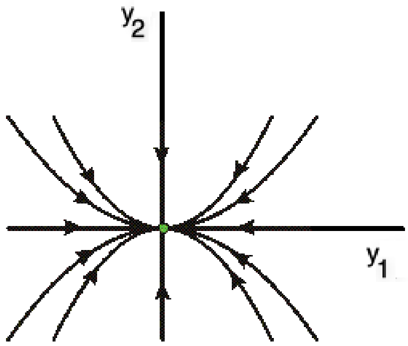
In a phase portrait, the state of the system is represented by a point, and the arrows indicate the direction of motion of the system at each point in the phase space. The trajectory of the system over time can be traced by following the arrows in the phase portrait.

Phase portraits are commonly used in physics, engineering, and other sciences to study the behavior of systems that change over time, such as oscillating springs, pendulums, and chemical reactions. They can also be used to analyze more complex systems, such as biological networks and economic models.

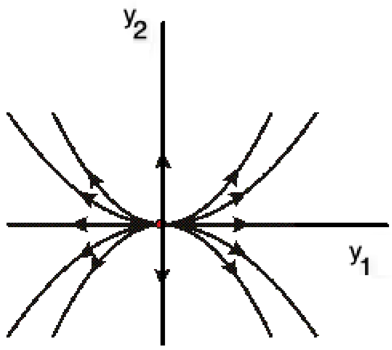
With t starting with 0 and $\rightarrow \infty$, we have the following classification.

An equilibrium point is stable if for some initial value close to the equilibrium point, the solution will eventually stay close to the equilibrium point. An equilibrium point is asymptotically stable if for some initial value close to the equilibrium point, the solution will converge to the equilibrium point.

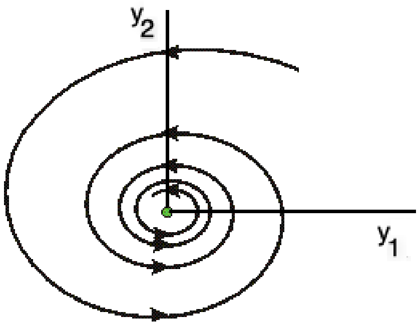
Eigenvalue λ_1, λ_2	Critical Point type	Stability as $t \to \infty$
real, distinct, -ve	Node	Stable
real, distinct, +ve	Node	Unstable
real, equal, -ve	Node	Stable
real, equal, +ve	Node	Unstable
real, distinct, opposite sign	Saddle	Unstable
complex with -ve real part	Spiral sink	Stable
complex with +ve real part	Spiral source	Unstable
complex with real part = 0	Center	Stable, not asymptotically



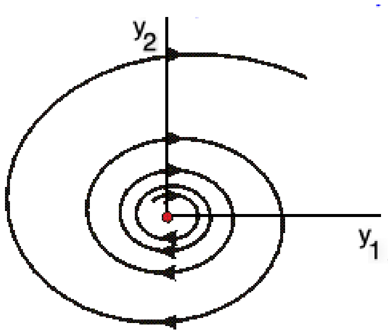
Node (stable, sink)



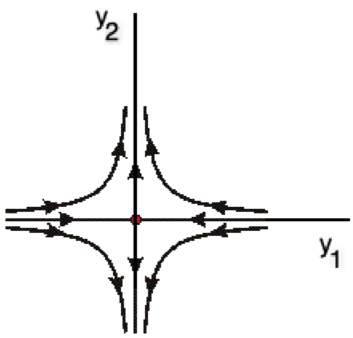
Node (unstable, source)



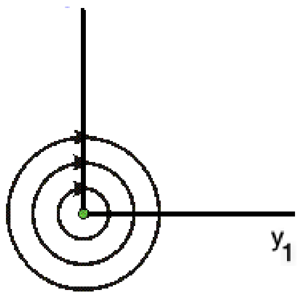
Spiral (stable)



Spiral (unstable)



Saddle Point



Center