Chapter 20
Waves and Sound

Waves move the form while leaving place intact,
A measured dance where time and space interact,
From whispering sound to light that spans the skies,
Law shapes the rhythm through which nature speaks and flies

Wave phenomena arise when a physical system supports oscillations that propagate through space and time. Mechanical waves require a material medium, while sound waves represent longitudinal mechanical disturbances perceived by the human ear. This chapter develops the mathematical description of waves with emphasis on sound.

20.1 Mechanical Waves

Mechanical waves result from the interaction between inertia and restoring forces in an elastic medium.

20.1.1 Wave Function and Displacement

Consider a one-dimensional medium extending along the x-axis. Let $y(x, t)$ denote the displacement of a particle from its equilibrium position at position x and time t.

The function

$$y : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

describes the wave motion.

20.1.2 Classification of Mechanical Waves

Mechanical waves are classified according to the direction of particle motion relative to wave propagation.

- Transverse waves, where displacement is perpendicular to propagation
- Longitudinal waves, where displacement is parallel to propagation

20.2 The One-Dimensional Wave Equation

20.2.1 Derivation of the Wave Equation

For small oscillations in a uniform elastic medium, the restoring force is proportional to the local curvature of the displacement. Applying Newton's second law to an infinitesimal element yields

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

where v is the speed of wave propagation in the medium.

20.2.2 General Solution

The general solution of the one-dimensional wave equation is given by d'Alembert's form

$$y(x,t) = f(x - vt) + g(x + vt)$$

where f and g represent waves traveling in opposite directions.

20.3 Boundary Conditions and Normal Modes

20.3.1 Fixed and Free Boundaries

For a string of length L fixed at both ends, the displacement must satisfy

$$y(0,t) = 0$$

$$y(L,t) = 0$$

These boundary conditions restrict the allowable wave solutions.

20.3.2 Normal Modes of a Stretched String

Solutions satisfying the fixed-end conditions are standing waves of the form

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t)$$

where n is a positive integer. The allowed angular frequencies are

$$\omega_n = \frac{n\pi v}{L}$$

The corresponding frequencies are

$$f_n = \frac{nv}{2L}$$

Each allowed pattern is called a normal mode of vibration.

20.4 Harmonic Waves

20.4.1 Sinusoidal Waveform

A sinusoidal or harmonic wave is described by

$$y(x,t) = A\sin(kx - \omega t + \phi)$$

where A is the amplitude, k is the wave number, ω is the angular frequency, and ϕ is the phase constant.

The wavelength λ and frequency f are related to k and ω by

$$k = \frac{2\pi}{\lambda}$$

$$\omega = 2\pi f$$

The wave speed satisfies

$$v = \frac{\omega}{k}$$

20.4.2 Energy Transport

The mechanical energy associated with a wave is distributed between kinetic and potential forms. The average power transmitted is proportional to the square of the amplitude

$$P \propto A^2$$

20.5 Superposition and Interference

Because the wave equation is linear, the principle of superposition holds. If y_1 and y_2 are solutions, then

$$y = y_1 + y_2$$

is also a solution.

Interference effects arise from phase differences between overlapping waves.

20.6 Standing Waves

Standing waves result from the superposition of two waves of equal amplitude and frequency traveling in opposite directions.

The resulting displacement may be written as

$$y(x,t) = 2A\sin(kx)\cos(\omega t)$$

Nodes occur at positions satisfying

$$kx = n\pi$$

where n is an integer.

20.7 Sound Waves

Sound waves are longitudinal mechanical waves that propagate through elastic media such as gases, liquids, and solids.

20.7.1 Longitudinal Displacement and Pressure

Let $\xi(x,t)$ denote the longitudinal displacement of particles in the medium. The pressure variation $p(x,t)$ is related to the displacement gradient by

$$p(x,t) = -B\frac{\partial \xi}{\partial x}$$

where B is the bulk modulus.

20.7.2 Wave Equation for Sound

The longitudinal displacement satisfies the wave equation

$$\frac{\partial^2 \xi}{\partial t^2} = v^2\frac{\partial^2 \xi}{\partial x^2}$$

The speed of sound is given by

$$v = \sqrt{\frac{B}{\rho}}$$

where ρ is the equilibrium density.

20.7.3 Intensity and Loudness

The intensity of sound is defined as the average power transmitted per unit area

$$I = \frac{P}{A}$$

Intensity is proportional to the square of the pressure amplitude

$$I \propto p_{\mathrm{max}}^2$$

20.8 Doppler Effect

The Doppler effect describes the change in observed frequency due to relative motion between the source and the observer.

For sound waves, the observed frequency is

$$f' = f\frac{v \pm v_o}{v \mp v_s}$$

where v_o is the observer velocity and v_s is the source velocity. The upper sign corresponds to motion toward each other.

20.9 Beats

Beats occur when two sound waves of slightly different frequencies interfere.

If the frequencies are f_1 and f_2, the beat frequency is

$$f_{\text{beat}} = |f_1 - f_2|$$

The resulting sound intensity varies periodically in time.

20.10 Closing Remarks

Boundary conditions determine the discrete normal modes of vibrating systems. Doppler shift and beats illustrate how wave frequency depends on motion and superposition. Together, these results complete the undergraduate description of mechanical and sound waves.