Chapter 23

Electricity & Magnetism

Charges rest and fields stand silently arranged,
Currents move and space itself is changed,
From shifting flux new forces intertwine,
One law binds light, charge, and time's design

Electric and magnetic phenomena originate from electric charge and its motion. Their unification is achieved through Maxwell's equations, which may be derived systematically from experimental laws and conservation principles.

23.1 Electric Field and Gauss's Law

23.1.1 From Coulomb's Law to Electric Field

Coulomb's law gives the force between two point charges

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$$

The electric field is defined as force per unit test charge

$$\mathbf{E} = \frac{\mathbf{F}}{q}$$

For a point charge q

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

23.1.2 Derivation of Gauss's Law

Consider a spherical surface of radius r centered on a point charge q. The electric field is radial and constant in magnitude over the surface.

The electric flux through the sphere is

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A}$$

Substituting the field magnitude

$$\Phi_E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}(4\pi r^2)$$

which simplifies to

$$\Phi_E = \frac{q}{\varepsilon_0}$$

For any closed surface enclosing charge Q_{enc}

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

Using the divergence theorem

$$\oint \mathbf{E} \cdot d\mathbf{A} = \int (\nabla \cdot \mathbf{E}) dV$$

yields the differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

This is Gauss's law for electricity.

23.2 Magnetic Field and Gauss's Law for Magnetism

23.2.1 Absence of Magnetic Monopoles

Experimental observation shows that magnetic field lines form closed loops and no isolated magnetic charges exist.

Therefore, the magnetic flux through any closed surface is zero

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$

Applying the divergence theorem gives

$$\nabla \cdot \mathbf{B} = 0$$

This is Gauss's law for magnetism.

23.3 Faraday's Law of Electromagnetic Induction

23.3.1 Induced Electric Fields

Faraday discovered that a changing magnetic flux induces an electromotive force

$$\mathscr{E} = -\frac{d\Phi_B}{dt}$$

The electromotive force around a closed loop is defined as

$$\mathscr{E} = \oint \mathbf{E} \cdot d\mathbf{l}$$

Equating the two expressions

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \oint \mathbf{B} \cdot d\mathbf{A}$$

Using Stokes' theorem

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{E}) \cdot d\mathbf{A}$$

leads to the differential form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This is Faraday's law in differential form.

23.4 Ampère's Law and Its Inconsistency

23.4.1 Original Ampère's Law

Ampère's circuital law for steady currents is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

Using Stokes' theorem

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

where $\mathbf{J}$ is the current density.

23.4.2 Conflict with Charge Conservation

Charge conservation is expressed by the continuity equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Taking the divergence of Ampère's law yields

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0$$

which implies

$$\nabla \cdot \mathbf{J} = 0$$

This contradicts the continuity equation for time-varying charge density.

23.5 Maxwell's Displacement Current

To restore consistency, Maxwell introduced an additional term called the displacement current.

The modified Ampère–Maxwell law is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Taking the divergence

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E})$$

Substituting Gauss's law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

yields the continuity equation, confirming consistency.

23.6 Maxwell's Equations

The complete set of Maxwell's equations in differential form is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{23.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{23.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{23.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{23.4}$$

23.7 Electromagnetic Waves

In free space, where $\rho = 0$ and $\mathbf{J} = 0$, Maxwell's equations reduce to wave equations.

For the electric field

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

The wave speed is

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

which equals the speed of light, revealing the electromagnetic nature of light.

23.8 Closing Remarks

Maxwell's equations emerge naturally from experimental laws combined with charge conservation and field continuity. Their derivation unifies electricity, magnetism, and optics within a single mathematical framework.