Chapter 26

Tensors

Tensors provide a mathematical framework for describing physical quantities that are independent of the choice of coordinate system. They generalize scalars and vectors and play a central role in continuum mechanics, electromagnetism, relativity, and modern physics.

26.1 Scalars and Vectors

26.1.1 Scalars

A scalar is a quantity that is fully described by a single number and remains unchanged under coordinate transformations.

Examples include mass, temperature, and energy.

26.1.2 Vectors

A vector is a quantity with magnitude and direction. In a Cartesian coordinate system, a vector **A** is represented by components

$$\mathbf{A} = (A^1, A^2, A^3)$$

Under a change of coordinates, the components transform linearly.

26.2 Coordinate Transformations

Consider two coordinate systems related by a linear transformation

$$x^{i'} = \frac{\partial x^{i'}}{\partial x^j} x^j$$

The transformation matrix is defined as

$$\Lambda^{i'}{}_j = \frac{\partial x^{i'}}{\partial x^j}$$

26.3 Definition of a Tensor

A tensor is defined by how its components transform under a change of coordinates.

26.3.1 Contravariant Vectors

A contravariant vector V^i transforms as

$$V^{i'} = \frac{\partial x^{i'}}{\partial x^j} V^j$$

26.3.2 Covariant Vectors

A covariant vector W_i transforms as

$$W_{i'} = \frac{\partial x^j}{\partial x^{i'}} W_j$$

26.4 Metric Tensor

The metric tensor defines distances and angles in a space.

In a general coordinate system, the squared line element is

$$ds^2 = g_{ij} dx^i dx^j$$

The metric tensor g_{ij} is symmetric

$$g_{ij} = g_{ji}$$

26.5 Raising and Lowering Indices

The metric tensor allows conversion between covariant and contravariant components.

Lowering an index is performed by

$$V_i = g_{ij} V^j$$

Raising an index is performed using the inverse metric g^{ij}

$$V^i = g^{ij} V_j$$

26.6 General Tensors

A tensor of type (p, q) has p contravariant indices and q covariant indices.

Its components transform as

$$T^{i'_1 \dots i'_p}{}_{j'_1 \dots j'_q} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \dots \frac{\partial x^{j_q}}{\partial x^{j'_q}} T^{i_1 \dots i_p}{}_{j_1 \dots j_q}$$

26.7 Tensor Operations

26.7.1 Addition and Scalar Multiplication

Tensors of the same type may be added componentwise.

26.7.2 Tensor Product

The tensor product of two tensors A and B produces a tensor of higher rank

$$(A \otimes B)^{ij} = A^i B^j$$

26.7.3 Contraction

Contraction reduces the rank of a tensor by summing over one upper and one lower index

$$T^i{}_i$$

26.8 Important Physical Tensors

26.8.1 Stress Tensor

In continuum mechanics, the stress tensor σ_{ij} relates force to area

$$F_i = \sigma_{ij} n^j$$

where n^j is the normal vector.

26.8.2 Moment of Inertia Tensor

The moment of inertia tensor is defined as

$$I_{ij} = \sum_k m_k (\delta_{ij} r_k^2 - x_{k,i} x_{k,j})$$

26.8.3 Electromagnetic Field Tensor

In relativistic electrodynamics, the electromagnetic field is represented by a rank-2 tensor.

26.9 Tensor Calculus

26.9.1 Partial Derivatives

The partial derivative of a tensor is generally not a tensor.

26.9.2 Covariant Derivative

To preserve tensorial character, the covariant derivative is introduced

$$\nabla_k V^i = \partial_k V^i + \Gamma^i_{kj} V^j$$

where Γ^i_{kj} are the Christoffel symbols.

26.10 Closing Remarks

Tensors provide a coordinate-independent language for physical laws. Their systematic use unifies diverse areas of physics and enables the formulation of laws valid in arbitrary coordinate systems.