Chapter 27

Differential Geometry

Differential geometry studies geometric structures using calculus. It provides the mathematical language for describing curves, surfaces, and manifolds, and forms the foundation of modern theories such as general relativity, continuum mechanics, and gauge field theory.

27.1 Manifolds

27.1.1 Definition of a Manifold

A smooth manifold $\mathscr{M}$ of dimension n is a topological space that locally resembles $\mathbb{R}^n$ and admits smooth coordinate charts.

Each point $p \in \mathscr{M}$ has a neighborhood with coordinates

$$(x^1, x^2, \ldots, x^n)$$

27.1.2 Coordinate Transformations

On overlapping coordinate charts, coordinates transform smoothly

$$x^{i'} = x^{i'}(x^1, \ldots, x^n)$$

Differential geometry studies structures that are invariant under such transformations.

27.2 Curves and Tangent Vectors

27.2.1 Curves on a Manifold

A curve on $\mathscr{M}$ is a smooth mapping

$$\gamma : \mathbb{R} \to \mathscr{M}$$

In local coordinates, the curve is represented by

$$x^i = x^i(t)$$

27.2.2 Tangent Vectors

The tangent vector to a curve at a point is defined as

$$v^i = \frac{dx^i}{dt}$$

The collection of all tangent vectors at a point forms the tangent space $T_p\mathscr{M}$.

27.3 Cotangent Space and Differential Forms

The dual space to the tangent space is the cotangent space $T_p^*\mathscr{M}$.

A differential one-form is written as

$$\omega = \omega_i dx^i$$

Differential forms provide a coordinate-independent framework for integration.

27.4 Metric Tensor

A metric tensor assigns an inner product to tangent vectors.

The squared line element is

$$ds^2 = g_{ij} dx^i dx^j$$

The metric tensor is symmetric

$$g_{ij} = g_{ji}$$

Distances and angles are defined through the metric.

27.5 Connection and Covariant Derivative

27.5.1 Parallel Transport

To compare vectors at different points, a notion of parallel transport is required.

This is achieved using a connection.

27.5.2 Covariant Derivative

The covariant derivative of a vector field V^i is defined as

$$\nabla_j V^i = \partial_j V^i + \Gamma^i_{jk} V^k$$

where Γ^i_{jk} are the Christoffel symbols.

For a covariant vector

$$\nabla_j V_i = \partial_j V_i - \Gamma^k_{ji} V_k$$

27.6 Geodesics

Geodesics are curves that generalize straight lines to curved spaces.

They extremize the path length

$$s = \int ds$$

The geodesic equation is

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

27.7 Curvature

27.7.1 Riemann Curvature Tensor

Curvature measures the non-commutativity of covariant derivatives.

The Riemann curvature tensor is defined by

$$R^i{}_{jkl} = \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^i_{km}\Gamma^m_{jl} - \Gamma^i_{lm}\Gamma^m_{jk}$$

27.7.2 Ricci Tensor and Scalar Curvature

The Ricci tensor is obtained by contraction

$$R_{ij} = R^k{}_{ikj}$$

The scalar curvature is

$$R = g^{ij}R_{ij}$$

27.8 Integration on Manifolds

The invariant volume element on a manifold is

$$dV = \sqrt{|g|}\, d^n x$$

where g is the determinant of the metric tensor.

27.9 Differential Geometry in Physics

Differential geometry provides the mathematical structure for physical laws:

- Geodesics describe free particle motion
- Curvature encodes gravitational effects
- Differential forms unify electromagnetic laws

27.10 Closing Remarks

Differential geometry replaces flat space intuition with intrinsic geometric structure. By expressing physical laws in coordinate-independent form, it provides a universal language for modern theoretical physics.